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Alfredo Raya[1] and Edward D Reyes[2]

[1] Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo,
Apartado Postal 2-82, CP 58040, Morelia, Michoacán, Mexico
[2] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Circuito Exterior s/n, Ciudad Universitaria, CP 04510, DF, Mexico

E-mail: raya@ifm.umich.mx

Abstract
Through an explicit calculation for a Lagrangian in quantum electrodynamics in (2+1)-spacetime dimensions (QED_3), making use of the relativistic Kubo formula, we demonstrate that the filling factor accompanying the quantized electrical conductivity for massive Dirac fermions of a single species in two spatial dimensions is a half (in natural units) when time reversal and parity symmetries of the Lagrangian are explicitly broken by the fermion mass term. We then discuss the most general form of the QED_3 Lagrangian, for both irreducible and reducible representations of the Dirac matrices in the plane, with emphasis on the appearance of a Chern–Simons term. We also identify the value of the filling factor with a zero field quantum Hall effect (QHE).

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1. Motivation

One of the most remarkable phenomena in condensed matter physics discovered in the last few decades of the past century is, indeed, the quantum Hall effect (QHE)—a striking manifestation of quantum mechanics at the macroscopic level. The experimental observation of QHE is the vanishingly small diagonal electrical conductivity $\sigma_{xx} \to 0$ of a bidimensional non-relativistic electron gas subject to a strong perpendicular magnetic field, while the off-diagonal conductivity is quantized according to

$$\sigma_{xy} = -\nu \frac{e^2}{2\pi} \tag{1}$$

in natural units ($\hbar = c = 1$), where e is the electron charge and ν is the so-called filling factor. This is defined as the ratio of the density of electrons in the sample to the magnetic

field strength, and can be a small integer, dubbed as integer QHE (IQHE) [1], or a fraction with odd denominator, the so-called fractional QHE (FQHE) [2]. In view of the lack of relativistic corrections [3, 4] to the quantization rule (1), a description of the QHE in terms of a full-fledged relativistic quantum field theory is highly desirable. Quantum electrodynamics in (2+1)-spacetime dimensions, QED$_3$, provides a natural framework for this purpose. This theory has useful applications, for example, in high-T$_c$ superconductivity [5] and in the field of dynamical chiral symmetry breaking, where it provides an attractive battleground for lattice and continuum studies [6]. More recently, QED$_3$ has been made use of in the description of the unconventional QHE in graphene [7]. This material consists of an isolated single atomic layer of graphite, an ideal realization of a bidimensional system, which exhibits an unusual half-integer QHE in which massless 'relativistic' carriers participate in the effect. A word of caution is at hand, since the term 'relativistic' in the effective description of condensed matter systems of this kind has nothing to do with the familiar Lorentz symmetry in (2+1)-dimensions of the high energy physics (HEP) realm. Such a symmetry is not present in systems like graphene simply because the corresponding Dirac Lagrangian contains the Fermi velocity, v_F, instead of the velocity of light, c. This explains the use of the quotation marks. Charge carriers in graphene are 'relativistic' in the sense that their energy–momentum dispersion relation is linear as opposed to the standard quadratic dispersion relation of non-relativistic systems. In this paper, however, we give an idealized description of the QHE in terms of QED$_3$, which, of course, afterwards would have to be adapted to the real physical systems with their observed continuous and discrete symmetries.

An ideal tool for the computation of the transverse conductivity, and thus of the corresponding filling factor, is the Kubo formula [8]. Its field theoretical analog in the non-relativistic case was introduced in [9] based upon gauge invariance in terms of the Ward–Green–Takahashi identities [10]. The filling factor is found through

$$\nu = \frac{1}{24\pi^2} \int d^3 p \, \epsilon_{\mu\rho\lambda} \text{Tr} \left[\partial^\mu S^{-1}(p) S(p) \partial^\rho S^{-1}(p) S(p) \partial^\lambda S^{-1}(p) S(p) \right], \quad (2)$$

where $S(p)$ is the electron propagator, $\epsilon_{\mu\rho\lambda}$ the Levi-Civita symbol and $\partial_\mu = \partial/\partial p^\mu$. The relativistic version of this expression preserves its functional form [4], thus allowing a microscopic description of the QHE in terms of relativistic Dirac fermions. Pursuing this aim, Acharya and Swamy (hereafter referred to as AS) established that QED$_3$ naturally leads to the IQHE [4]. They considered non-interacting electrons, i.e., particles whose (inverse) propagator is simply given by

$$S^{-1}(p) = \not{p} - m, \tag{3}$$

where m (assumed to be a positive quantity) is the effective mass of the electrons and they choose the following irreducible representation of the γ-matrices in terms of the Pauli matrices:

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^2, \tag{4}$$

which verify $\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\lambda}\gamma_\lambda$ with $\gamma_\mu = g_{\mu\nu}\gamma^\nu$. This model has useful applications in the description of one-dimensional polyacetylene [11]. In its full quantum field theoretical form, it exhibits the so-called parity anomaly [12], which has a condensed matter realization in a model which describes QHE without Landau levels [13]. Inserting the propagator (3) into equation (2), with the aid of the properties of the Dirac matrices,

$$\mathrm{Tr}\,[\gamma^\mu] = 0$$
$$\mathrm{Tr}\,[\gamma^\mu \gamma^\nu] = 2g^{\mu\nu}$$
$$\mathrm{Tr}\,[\gamma^\mu \gamma^\nu \gamma^\lambda] = -2\mathrm{i}\epsilon^{\mu\nu\lambda} \tag{5}$$
$$\mathrm{Tr}\,[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma] = 2(g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\sigma})$$
$$\mathrm{Tr}\,[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\rho] = -2\mathrm{i}(g^{\mu\nu}\epsilon^{\lambda\sigma\rho} + g^{\lambda\sigma}\epsilon^{\mu\nu\rho} + g^{\sigma\rho}\epsilon^{\mu\nu\lambda} - g^{\lambda\rho}\epsilon^{\mu\nu\sigma}),$$

AS find

$$
\begin{aligned}
\nu &= \frac{1}{24\pi^2} \int \mathrm{d}^3 p\, \epsilon_{\mu\rho\lambda} \mathrm{Tr}\left[\gamma^\mu \left(\frac{\not{p}+m}{p^2-m^2} \right) \gamma^\rho \left(\frac{\not{p}+m}{p^2-m^2} \right) \gamma^\lambda \left(\frac{\not{p}+m}{p^2-m^2} \right) \right] \\
&= \frac{1}{24\pi^2} \int \frac{\mathrm{d}^3 p}{(p^2-m^2)^3} \epsilon_{\mu\rho\lambda} \mathrm{Tr}\,[\gamma^\mu \not{p}\gamma^\rho \not{p}\gamma^\lambda \not{p} + m^3 \gamma^\mu \gamma^\rho \gamma^\lambda \\
&\quad + m(\gamma^\mu \not{p}\gamma^\rho \gamma^\lambda \not{p} + \gamma^\mu \gamma^\rho \not{p}\gamma^\lambda \not{p} + \gamma^\mu \not{p}\gamma^\rho \not{p}\gamma^\lambda) \\
&\quad + m^2(\gamma^\mu \gamma^\rho \gamma^\lambda \not{p} + \gamma^\mu \not{p}\gamma^\rho \gamma^\lambda + \gamma^\mu \gamma^\rho \not{p}\gamma^\lambda)] \\
&= \frac{1}{24\pi^2} \int \mathrm{d}^3 p\, \frac{12\mathrm{i}m(p^2-m^2)}{(p^2-m^2)^3} \\
&= \frac{\mathrm{i}m}{2\pi^2} \int \mathrm{d}^3 p\, \frac{1}{(p^2-m^2)^2} = -\frac{1}{2}, \tag{6}
\end{aligned}
$$

where the result in the last line comes after a Wick rotation to Euclidean space and a standard integration. Perhaps because this value of the filling factor does not correspond to the IQHE, nor to the FQHE, AS tried to give sense to their findings making three crucial statements:

(i) because electrons possess 2 spin states, a factor of 2 should be put by hand in order to obtain the IQHE.

(ii) The trace in equation (2) would have vanished for the usual QED$_4$ matrices.

(iii) The Chern–Simons term plays no role whatsoever.

We find these statements misleading and, as such, these cast doubts on the results of works based on them [14, 15]. Although we agree with AS in that the presence of the electron mass is crucial for QED$_3$ exhibiting ordinary QHE (and not the unconventional QHE found in graphene), we shall argue that these three statements cannot be correct. Criticisms to the 'factor of 2' have already appeared in the literature [16, 17]. Yet, our approach, based upon the analysis of the discrete symmetries (charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, which for us is the inversion of one spatial axis, and time reversal $\mathcal{T}$) of the QED$_3$ Lagrangian, offers a new understanding of the problems of these mishaps.

We have organized this paper as follows: in section 2 we discuss why the result in equation (6) is in fact correct, by analyzing the Dirac particle spectrum when irreducible representations of γ-matrices are chosen [18, 19]. Section 3 is devoted to the calculation of the filling factor ν making use of the reducible representation of the usual γ-matrices of QED$_4$, considering all types of fermion masses which can be present in (2+1)-dimensions. We discuss the most general form of the QED$_3$ Lagrangian, with emphasis on the discrete symmetries ($\mathcal{C}, \mathcal{P}, \mathcal{T}$) transformation properties of mass terms, both for electrons and photons (Chern–Simons term) in section 4, and discuss how these can radiatively induce each other. Finally, in section 5 we identify the half-filling factor state for a single electron species as a zero magnetic field QHE and then summarize our findings.

2. Irreducible Dirac fermions

We start by noting that the fermion propagator in equation (3) can be read off from the (2+1)-analog of the usual free Dirac Lagrangian in 4D:

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} - m)\psi, \tag{7}$$

with the γ-matrices given in equation (4). The spectrum of solutions of the resulting Dirac equation, which completes the corresponding Hilbert space, in the sense that the completeness relations $\sum u\bar{u} = \slashed{p} + m$ and $\sum v\bar{v} = \slashed{p} - m$ are fulfilled, is

$$
\begin{aligned}
\psi^P(x) &= \begin{pmatrix} 1 \\ \frac{p_x - ip_y}{E+m} \end{pmatrix} e^{-ix\cdot p} \equiv u(p)\, e^{-ix\cdot p} \\
\psi^N(x) &= \begin{pmatrix} \frac{p_x + ip_y}{E+m} \\ 1 \end{pmatrix} e^{ix\cdot p} \equiv v(p)\, e^{ix\cdot p},
\end{aligned}
\tag{8}
$$

and consists of only a positive energy solution (particle spinor with spin up), and a negative energy one (antiparticle spinor with spin down) [18]. Yet, these solutions fail to incorporate various symmetries of the ordinary Dirac spectrum in 4D. For example, the solutions, and correspondingly the Lagrangian in its *inherited* form (7), are not invariant under a Parity transformation $\mathcal{P}$—which for consistency with Lorentz symmetry corresponds to the inversion of one spatial axis, nor under a time reversal transformation $\mathcal{T}$. Furthermore, only one out of the two spin states of the physical electrons is present in (2+1)-dimensions if we consider an irreducible representation for the Dirac matrices. This fact makes it clear that the result in equation (6) is, in fact, correct and there is no need to put by hand the factor of 2 advocated in [12] and made use of in [4, 14, 15]. One might argue that since in condensed matter physics spin plays the role of flavor in HEP, one should in fact put the spin factor of 2 by hand. However, one cannot simply push this argument to the idealized Lagrangian of QED$_3$ we are considering here. We shall postpone the discussion of the half-filling state, and content ourselves at this stage with the filling factor vanishing or not.

The two spin states and symmetry features of the familiar spectrum of solutions to the Dirac equation in 4D can be recovered owing to the fact that in (2+1)-dimensions there exists a second irreducible representation of the Dirac matrices in terms of the Pauli matrices, given by

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = -i\sigma^2, \tag{9}$$

with the property $\gamma^\mu \gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma_\lambda$. Solutions of the Dirac equation in this representation are

$$
\begin{aligned}
\psi^P(x) &= \begin{pmatrix} \frac{p_x + ip_y}{E+m} \\ 1 \end{pmatrix} e^{-ix\cdot p} \equiv u(p)\, e^{-ix\cdot p} \\
\psi^N(x) &= \begin{pmatrix} 1 \\ \frac{p_x - ip_y}{E+m} \end{pmatrix} e^{ix\cdot p} \equiv v(p)\, e^{ix\cdot p},
\end{aligned}
\tag{10}
$$

and correspond to particle spinor with spin down and antiparticle with spin up [18]. These solutions fulfil the completeness relations $\sum u\bar{u} = \slashed{p} + m$ and $\sum v\bar{v} = \slashed{p} - m$, but also present the lack of spin states and symmetry properties of the familiar 4D solutions. Nevertheless, taking into account solutions for both representations, (4) and (9), attaching to them labels A and B, respectively, we recover the features of the ordinary Dirac spectrum, namely, two spin states for the electrons and their corresponding Lorentz conjugated positron states. The

two 'irreducible' fermion fields can be cast into the following *extended* form of the free Dirac Lagrangian [18, 19][3]:

$$\mathcal{L} = \bar{\psi}_A(i\slashed{\partial} - m)\psi_A + \bar{\psi}_B(i\slashed{\partial} + m)\psi_B. \tag{11}$$

As we noted before, neither under $\mathcal{P}$ nor under $\mathcal{T}$, the fields ψ_A and ψ_B transform onto themselves. In fact, under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations, these fields transform as

$$
\begin{aligned}
(\psi_A)^{\mathcal{C}} &= \gamma^2 e^{i\eta_1}(\bar{\psi}_A)^T & (\psi_B)^{\mathcal{C}} &= \gamma^2 e^{i\eta_2}(\bar{\psi}_B)^T \\
(\psi_A)^{\mathcal{P}} &= -i\gamma^1 e^{i\phi_1}(\psi_B) & (\psi_B)^{\mathcal{P}} &= -i\gamma^1 e^{i\phi_2}(\psi_A) \\
(\psi_A)^{\mathcal{T}} &= i\gamma^3 e^{i\varphi_1}(\bar{\psi}_B)^T & (\psi_B)^{\mathcal{P}} &= -i\gamma^3 e^{i\varphi_2}(\bar{\psi}_A)^T,
\end{aligned}
\tag{12}
$$

where η_i, ϕ_i and φ_i, $i = 1, 2$, are constant phases. This shows that the extended Lagrangian (11) is $\mathcal{CPT}$ invariant [19]. The model (11) has recently been considered to study the formation of $\bar{\psi}\psi$-condensates in the presence of magnetic fields even in the absence of fermion masses [20]. For this Lagrangian, the fermion propagators are of the form (3), but now m alternates in sign between ψ_A and ψ_B. Thus, since there are two fermion species in the Lagrangian (11), the filling factor is [21]

$$\nu = -\frac{1}{2} \sum_\alpha \text{sgn}(m_\alpha) \tag{13}$$

where m_α is the mass of fermion species α, and thus vanishes in this case. This result is understandable because σ_{xy} is a $\mathcal{P}$ and $\mathcal{T}$ violating quantity, whereas the Lagrangian in equation (11), from which we are deriving it, is not. Thus σ_{xy}, or equivalently ν, can only be zero. This will be further clarified in section 4.

The presence of two irreducible fermion fields in (11) naturally suggests that these can be merged into one reducible four-component spinor and hence we can make use of the ordinary 4×4 Dirac matrices of QED$_4$. Nevertheless, care should be taken since, in (2+1)-dimensions, further mass terms, besides the ordinary $m\bar{\psi}\psi$, can arise. Such an issue is discussed below.

3. Reducible Dirac fermions

If ordinary 4×4 Dirac matrices are made use of, only three of them are required to describe the dynamics of electrons in (2+1)-dimensions, for example $\{\gamma^0, \gamma^1, \gamma^2\}$, which can be represented as

$$\gamma^0 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{14}$$

In that case, we have two other γ-matrices which commute with all the three matrices above, in such a fashion that the corresponding massless Dirac Lagrangian is invariant under the chiral-like transformations $\psi \rightarrow e^{i\alpha\gamma^3}\psi$, and $\psi \rightarrow e^{i\alpha\gamma^5}\psi$, that is, it is invariant under a global $U(2)$ symmetry with generators 1, γ^3, γ^5 and $[\gamma^3, \gamma^5]$. Here

$$\gamma^3 = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{15}$$

I being the 2×2 unit matrix. This symmetry is broken by an ordinary mass term $m_e\bar{\psi}\psi$. But there exists another mass term (sometimes referred to as Haldane mass term) which is invariant under the 'chiral' transformations

$$m_o\bar{\psi}\frac{i}{2}[\gamma^3, \gamma^5]\psi \equiv m_o\bar{\psi}(i\tau)\psi. \tag{16}$$

---

[3] Note that only one irreducible representation of the Dirac matrices, say (4) is used.

If we write the 4-spinor as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{17}$$

we observe that under $\mathcal{P}$ and $\mathcal{T}$, the upper and lower components of this spinor transform, up to a phase, as [22]

$$
\begin{aligned}
(\psi_1(t, x, y))^{\mathcal{P}} &\to \sigma^1 \psi_2(t, -x, y) & (\psi_2(t, x, y))^{\mathcal{P}} &\to \sigma^1 \psi_1(t, -x, y) \\
(\psi_1(t, x, y))^{\mathcal{T}} &\to \sigma^2 \psi_2(-t, x, y) & (\psi_2(t, x, y))^{\mathcal{T}} &\to \sigma^2 \psi_1(-t, x, y).
\end{aligned}
\tag{18}
$$

Thus, the term $m_e \bar{\psi} \psi$ is even under each of these transformations, but $m_o \bar{\psi}(i\tau)\psi$ is not, although it is $\mathcal{PT}$ and thus $\mathcal{CPT}$ symmetric. Here the evenness and oddness of the mass terms under $\mathcal{P}$ and $\mathcal{T}$ justify the use of the subscripts 'e' and 'o'. The corresponding Euclidean space free *reducible* Dirac Lagrangian in this case has the form

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} - m_e - m_o \tau)\psi, \tag{19}$$

where the Euclidean Dirac matrices are chosen as

$$
\gamma^0 = \begin{pmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \qquad
\gamma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \qquad
\gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix},
\tag{20}
$$

such that

$$
\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad
\gamma^5 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad
\tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\tag{21}
$$

There are many planar condensed matter models in which the low energy sector can be written as this effective form of QED$_3$, for which the physical origin of the masses depends on the underlying system [23]; for example d-wave cuprate superconductors [5], d-density-wave states [24] and layered graphite [25], including graphene in the massless version [7]. The reader should bear in mind that the discrete $\mathcal{P}$ and $\mathcal{T}$ symmetries discussed above do not have direct relation to corresponding symmetries in two-dimensional condensed matter systems. For example, there is no symmetry with respect to a reflection of one spatial coordinate in graphene. Instead the space-inversion symmetry contains reflection of signs of two spatial coordinates and the exchange of the A and B atoms of the honeycomb lattice and $\mathbf{K}_\pm$ points in the Brillouin zone. On the other hand, the time reversal operation, which flips the spin signs, interchanges $\mathbf{K}_\pm$, but not sublattices.

In order to perform a calculation of the filling factor, it is convenient to introduce the chiral-like projectors

$$\chi_\pm = \tfrac{1}{2}(1 \pm \tau), \tag{22}$$

which verify [26] $\chi_\pm^2 = \chi_\pm$, $\chi_+ \chi_- = 0$, $\chi_+ + \chi_- = I$. The 'right handed' ψ_+ and 'left handed' ψ_- fermion fields in this case are given by $\psi_\pm = \chi_\pm \psi$. The $\chi_\pm$ project the upper and lower two component spinors (fermion species) out of the four-component ψ. The chiral-like decomposition of the free fermion propagator is

$$
\begin{aligned}
S(p) &= -\left(\frac{\slashed{p} + m_+}{p^2 + m_+^2} \chi_+ + \frac{\slashed{p} + m_-}{p^2 + m_-^2} \chi_- \right) \\
&\equiv -(S_+(p)\chi_+ + S_-(p)\chi_-),
\end{aligned}
\tag{23}
$$

where $m_\pm = m_e \pm m_o$. Hence, the right- and left-handed projections of the filling factor can be obtained from equation (2) as

$$\nu_\pm = \frac{1}{24\pi^2} \int d^3p \, \epsilon_{\mu\nu\lambda} \mathrm{Tr}\left[\partial^\mu S_\pm^{-1} S_\pm \partial^\rho S_\pm^{-1} S_\pm \partial^\lambda S_\pm^{-1} S_\pm \chi_\pm \right], \tag{24}$$

where we have omitted the dependence of the propagators on the fermion momentum p to avoid cumbersome notation. With the aid of the relations

$$
\begin{aligned}
&\mathrm{Tr}\,[\chi_\pm] = 2, \\
&\mathrm{Tr}\,[\gamma^\mu \chi_\pm] = 0, \\
&\mathrm{Tr}\,[\gamma^\mu \gamma^\nu \chi_\pm] = -2\delta^{\mu\nu}, \\
&\mathrm{Tr}\,[\gamma^\mu \gamma^\nu \gamma^\alpha \chi_\pm] = \mp 2\epsilon^{\mu\nu\alpha}, \\
&\mathrm{Tr}\,[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \chi_\pm] = 2(\delta^{\mu\nu}\delta^{\alpha\beta} + \delta^{\mu\beta}\delta^{\nu\alpha} - \delta^{\mu\alpha}\delta^{\nu\beta}),
\end{aligned}
\tag{25}
$$

the calculation of (24) becomes very similar to that in equation (6) :

$$
\begin{aligned}
\nu_\pm &= \frac{\epsilon_{\mu\rho\lambda}}{24\pi^2} \int \mathrm{d}^3 p\, \mathrm{Tr}\left[\gamma^\mu \left(\frac{\not p + m_\pm}{p^2 + m_\pm^2}\right) \gamma^\rho \left(\frac{\not p + m_\pm}{p^2 + m_\pm^2}\right) \gamma^\lambda \left(\frac{\not p + m_\pm}{p^2 + m_\pm^2}\right) \chi_\pm\right] \\
&= \frac{\epsilon_{\mu\rho\lambda}}{24\pi^2} \int \frac{\mathrm{d}^3 p}{\left(p^2 + m_\pm^2\right)^3} \mathrm{Tr}\big[\gamma^\mu \not p \gamma^\rho \not p \gamma^\lambda \not p \chi_\pm + m_\pm^3 \gamma^\mu \gamma^\rho \gamma^\lambda \chi_\pm \\
&\quad + m_\pm(\gamma^\mu \not p \gamma^\rho \gamma^\lambda \not p + \gamma^\mu \gamma^\rho \not p \gamma^\lambda \not p + \gamma^\mu \not p \gamma^\rho \not p \gamma^\lambda)\chi_\pm \\
&\quad + m_\pm^2(\gamma^\mu \gamma^\rho \gamma^\lambda \not p + \gamma^\mu \not p \gamma^\rho \gamma^\lambda + \gamma^\mu \gamma^\rho \not p \gamma^\lambda)\chi_\pm\big] \\
&= \mp \frac{1}{24\pi^2} \int \mathrm{d}^3 p\, \frac{12 m_\pm\left(p^2 + m_\pm^2\right)}{\left(p^2 + m_\pm^2\right)^3} \\
&= \mp \frac{m_\pm}{2\pi^2} \int \mathrm{d}^3 p\, \frac{1}{\left(p^2 + m_\pm^2\right)^2} = \mp \frac{1}{2}\mathrm{sgn}(m_\pm).
\end{aligned}
\tag{26}
$$

This proves the usefulness of the chiral-like projectors. Note that these projectors only account for the same degrees of freedom in a different basis.

From the above expressions, we obtain that

$$
\nu = \nu_+ + \nu_- = -\tfrac{1}{2}\mathrm{sgn}(m_+) + \tfrac{1}{2}\mathrm{sgn}(m_-).
\tag{27}
$$

This gives $\nu = 0$ if $m_o = 0$, and $\nu = -1$ if $m_e = 0$, $-1/2$ coming from each fermion species. Two comments are in order at this point: first, we have seen that even with the use of ordinary γ-matrices, the trace in equation (24) yields a nonvanishing result, contrary to the claims of AS [4]. This is so because of the presence of the mass term (16), which violates $\mathcal{P}$ and $\mathcal{T}$ just like the ordinary fermion mass term in the irreducible Dirac Lagrangian (7). Not surprisingly, this result comes about since the physical content of equations (2) and (24) is the same, regardless of the representation of the γ-matrices. Second, since in the reducible representation of the Dirac matrices two different species of electron fields are taken into account, each contributing with $-1/2$ to the filling factor, QED$_3$ with its full-fledged relativistic symmetry naturally leads to the IQHE. The reader should be careful in the interpretation of this result. As we previously pointed out, in condensed matter physics spin plays the role of flavor in HEP and two-dimensional Dirac fermions, combined into a four-dimensional Dirac spinor, describe gapless excitations near different nodal points of the Fermi surface, which in graphene, for example, belong to the $\mathbf{K}_\pm$ points in the extended Brillouin zone. In the absence of a magnetic field, one in fact has to take the spin factor 2 by hand. In the case we are considering here, such a factor comes from the flavor of fermions.

Note that it is only the $\mathcal{P}$ and $\mathcal{T}$ odd mass terms, both in the irreducible (7) and reducible (19) Lagrangians which leads to a nonvanishing quantized conductivity, in agreement with the general properties required for a theory to exhibit QHE [27]. This fact is important because of the Coleman–Hill theorem [28], which states that such a mass term radiatively gives

rise to a Chern–Simons term only at the one-loop level. So, it is desirable that such a term appears directly in the bare Lagrangian. We shall now discuss the most general Lagrangian of QED$_3$ under this observation and we argue why the third comment of AS cannot be correct either.

4. QED$_3$ Lagrangian

To start with the discussion of the QED$_3$ Lagrangian, let us recall that its 4D analog is of the form

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\gamma} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Int}}, \tag{28}$$

where

$$\mathcal{L}_{\gamma} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{29}$$

$\mathcal{L}_{\text{GF}}$ is the gauge fixing Lagrangian, whose form is irrelevant in our discussion, and $\mathcal{L}_{\text{Int}}$ is the interaction term which minimally couples matter fields to the photon field. This form, however, is not the most general in (2+1)-dimensions. Firstly, as we have seen in previous sections, we can work with reducible and irreducible representations for the γ-matrices. If we insist on sticking to a single irreducible representation, like the one in equation (4), the ordinary mass term $m\psi\bar{\psi}$ breaks both the $\mathcal{P}$ and $\mathcal{T}$ invariance of $\mathcal{L}_{\text{Dirac}}$. Furthermore, because of the Coleman–Hill theorem [28], this mass term (and any other odd mass term) would radiatively generate a Chern–Simons term (CST)

$$\mathcal{L}_{CS} = \frac{\vartheta}{4} \epsilon_{\mu\rho\lambda} F^{\mu\rho} A^{\lambda} \tag{30}$$

into the Lagrangian of QED$_3$, which induces a gauge invariant topological mass for the photon. In order to see that, consider the Lagrangian (28) with $\mathcal{L}_{\text{Dirac}}$ given in (7). The first radiative correction to the current–current correlation function, out of which one can also compute the transverse conductivity (see appendix), and that in the HEP language is known as the vacuum polarization, is given by

$$\Pi_{\mu\rho}(q) = \mathrm{i}e^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, \mathrm{Tr}\left[\gamma^{\mu} \frac{1}{\not{k} - m} \gamma_{\rho} \frac{1}{\not{k} + \not{q} - m} \right]. \tag{31}$$

One can see from the properties of the Dirac matrices (5) that the tensor structure of the vacuum polarization is

$$\begin{aligned}
\Pi_{\mu\rho}(q) &= \Pi^e_{\mu\rho}(q) + \Pi^o_{\mu\rho}(q) \\
&= \left(g_{\mu\rho} - \frac{q_\mu q_\rho}{q^2} \right) \Pi_e(q^2) + \mathrm{i}m\epsilon_{\mu\rho\lambda} q^{\lambda} \Pi_o(q^2),
\end{aligned} \tag{32}$$

where we have used once more the labels e and o for the even and odd parts of the vacuum polarization under $\mathcal{P}$ and $\mathcal{T}$ transformations and the factor m is used for later convenience. The tensor structure of the second term, which emerges from the trace of three γ-matrices, induces a Chern–Simons interaction $\epsilon_{\mu\rho\lambda} F^{\mu\rho} A^{\lambda}$, which at the level of Lagrangians corresponds to (30). A similar argument follows for $\mathcal{L}_{\text{Dirac}}$ of the form (19). As we shall see in the appendix, the linear part of $\Pi_o(q^2)$ is related to the Hall conductivity and thus to the filling factor. With the $\mathcal{P}$ and $\mathcal{T}$ preserving Lagrangian (11), the vacuum polarization has only the even tensor structure, i.e., $\Pi_o(q^2) \equiv 0$, which readily implies $\sigma_{xy} = 0$ and thus $\nu = 0$.

The CST also violates $\mathcal{P}$ and $\mathcal{T}$, but is $\mathcal{PT}$ and thus $\mathcal{CPT}$ invariant (see, for example, [29]). The converse is also true, if we start from a Lagrangian of QED$_3$ which includes the CST but no $\mathcal{P}$ and $\mathcal{T}$ violating fermion mass term, the CST would radiatively generate it

[30], in such a fashion that fermions acquire their usual relativistic energy spectrum, with a mass gap. Observe that in the case of the Dirac Lagrangian (11), no CST would be radiatively induced, because the two fermion fields appear with different signs for their mass terms. Thus, depending upon the choice of the representation for the Dirac matrices, we get three possible choices for the QED$_3$ Lagrangian :

- *Case I :* for the *inherited* Dirac Lagrangian (7),

$$\mathcal{L}^I_{\mathrm{QED}_3} = \bar{\psi}(\mathrm{i}\slashed{\partial} - m)\psi + \mathcal{L}_\gamma + \mathcal{L}_{GF} + \mathcal{L}_{\mathrm{Int}} + \frac{\vartheta}{4}\epsilon_{\mu\nu\lambda}A^\mu F^{\nu\lambda}. \tag{33}$$

  In this case, the mass term and the CST induce each other mutually. This is the most general form of the Lagrangian if the representation (4) or (9) is chosen.

- *Case II :* for the *extended* Dirac Lagrangian (11),

$$\mathcal{L}^E_{\mathrm{QED}_3} = \bar{\psi}_A(\mathrm{i}\slashed{\partial} - m)\psi_A + \bar{\psi}_B(\mathrm{i}\slashed{\partial} + m)\psi_B + \mathcal{L}_\gamma + \mathcal{L}_{GF} + \mathcal{L}^{A,B}_{\mathrm{Int}}. \tag{34}$$

  In this case there is no CST, because the signs of the fermion mass terms cancel between them any contribution to (30).

- *Case III :* for the *reducible* Dirac Lagrangian (19),

$$\mathcal{L}^R_{\mathrm{QED}_3} = \bar{\psi}(\mathrm{i}\slashed{\partial} - m_e - m_o\tau)\psi + \mathcal{L}_\gamma + \mathcal{L}_{GF} + \mathcal{L}_{\mathrm{Int}} + \frac{\vartheta}{4}\epsilon_{\mu\nu\lambda}A^\mu F^{\nu\lambda}. \tag{35}$$

  Here m_o and the CST induce each other mutually.

Thus, as we have seen in the previous sections, only those $\mathcal{P}$ and $\mathcal{T}$ violating Lagrangians, (33) and (35), give nonvanishing filling factors. Fermion masses which break $\mathcal{P}$ and $\mathcal{T}$ are responsible for such contributions, and these and the CST generate each other radiatively. Then, the sole presence of m in (33) or m_o in (35) unveils the presence of the CST, a statement that contradicts the third comment of AS. Below we discuss the consequences of the symmetries of these Lagrangians and their connection to the nonvanishing filling factor.

5. Summary and discussion

When and if the Lagrangian for electrons, restricted to live in a plane, is not invariant under the discrete symmetries of $\mathcal{P}$ and $\mathcal{T}$, a non-vanishing transverse conductivity develops and QHE emerges [27]. In this fashion, IQHE takes place as a result of the quantization conditions for individual electrons in a magnetic field. The number of filled Landau levels, the filling factor ν, is related to the induced effective action for the gauge field, namely, the CST, in the vacuum of the electrons. If interactions between electrons play a role, bulk effects are manifest, setting the scene for the FQHE to take place. In this effect, because of interactions between electrons, Landau levels are filled only partially. This can be visualized as if electrons were subject to an effective magnetic field, the external plus a fictitious field arising from their many-body wavefunction statistical properties. In the particular case when Landau levels are half filled, such an effective magnetic field vanishes. Thus, there is a duality between a system of electrons in a magnetic field at $\nu = 1/2$ and a system of non-interacting electrons which exhibits QHE at the zero magnetic field [17, 31]. As such, this duality can be exploited to obtain physical properties in a simpler manner.

For example, it was experimentally demonstrated that in a usual quantum Hall system, the $\nu = 1/2$ state is realized either in a wide single quantum well or a double quantum well [32]. Resonance of a surface sound wave with cyclotron orbits of charge carriers was also observed in this system, although perhaps the most striking observation was that this system develops a well-defined Fermi surface [33]. The theoretical explanation for the formation of such a Fermi

surface and other features observed in this state was proposed [34] precisely making use of this duality. Electrons in this case were considered non-relativistic in nature. The physical picture is that this system is an electrical dipole created by electrons and vortices [35]. But also models of massive relativistic Dirac fermions have been considered as a realization of this duality. The role of a Dirac mass term of the $\mathcal{P}$ and $\mathcal{T}$ violating type (Haldane mass) was first studied on the lattice to explain a QHE without Landau levels [13]. Later, such a mass was shown to be important to explain the nature of transitions in the IQHE [36, 21], playing the role of the energy in the usual treatment of the IQHE. In all these cases, the filling factor associated with massive Dirac fermions of a single species was found to be a half, in agreement with our findings. There is no problem with that because a single Dirac fermion is incapable of reproducing the behavior of a system with a finite number of degrees of freedom per volume. Lattice results suggest that there are always an even number of particles participating in the effect, at least one of them being a massive spectator [21]. More recently, the $\nu = 1/2$ result of the zero field QHE has been nicely explained in geometrical terms [17], identifying the corresponding conductivity with a solid angle, clarifying the problems with the 'factor of 2' of AS.

The identification of the discrete symmetries of the QED$_3$ Lagrangian, related to the different representations which can be given to the Dirac matrices and the appearance of the CST, sets the basis of a complete understanding of the system of non-interacting fermions which exhibits QHE without Landau levels. The duality between systems of interacting and non-interacting fermions, in the light of the emergence of 'relativistic' condensed matter systems, brings winds of optimism to the solution of current puzzles, like the minimal conductivity in graphene [37].

In summary, we have undertaken an idealized microscopical description of the QHE in a full-fledged relativistic quantum field theory, QED$_3$, by means of the field theoretical version of the Kubo formula, equation (2), written in terms of the fermion propagator. We have observed that this theory naturally leads to a half-filling QHE per fermion species present in the underlying Lagrangian, when $\mathcal{P}$ and $\mathcal{T}$ are explicitly broken by a fermion mass term. In the process, we have examined three critical statements made by AS in [4] from a Lagrangian point of view. We have found that for a single fermion species:

- there is no need to multiply this result by 2.
- The same result can be obtained considering the ordinary Dirac matrices, but considering the usual fermion mass term and a $\mathcal{P}$ and $\mathcal{T}$ violating mass term, equation (16).
- CST is implicit in the result.

Indeed, if an irreducible representation of the γ-matrices is made use of, (i) the ordinary mass term $m\bar{\psi}\psi$ is odd under $\mathcal{P}$ and $\mathcal{T}$; and (ii) the ordinary spectrum of solutions of the Dirac equation, although complete—in the sense that completeness relations are fulfilled—, lacks two polarization states for the electron [18]. One of the most significant features of the ordinary spectrum in 4D, namely, the existence of two spin states for electrons and positrons, can be recovered owing to the fact that there exists a second irreducible representation of the Dirac matrices which enforces us to consider two fermion species in an 'extended' Lagrangian [18, 19], equation (11); however, this yields a vanishing filling factor. Also, merging two different fermion species, making use of a reducible representation for the Dirac matrices, the Kubo formula yields a half-filling factor per species, provided a $\mathcal{P}$ and $\mathcal{T}$ violating mass term, equation (16), is included. Thus, we confirm that in order for a theory to exhibit QHE, its underlying Lagrangian must be $\mathcal{T}$ (and $\mathcal{P}$) violating, but $\mathcal{CPT}$ preserving [17, 27, 31]. In our case, the fermion mass terms, m in (7) and m_o in (19), are responsible for this. In the case of graphene, where massless Dirac fermions participate in the effect, the external magnetic

field is responsible for breaking the discrete symmetries. The vanishing of the filling factor in graphene in the zero field limit has been shown by lattice simulations [13] and by continuum studies [38]. Furthermore, a $\mathcal{P}$ and $\mathcal{T}$ violating fermion mass term radiatively generates a CST [28] and vice versa [30], rendering the energy spectrum of massive Dirac fermions in its usual relativistic (gapped) form. Thus, the sole presence of such a fermion mass implies the presence of the CST in the underlying Lagrangian. The existence of the CST implies a Hall conductivity, in this case characterized by a filling factor $\nu = 1/2$ per species. The non-integer nature of this ν suggests its origin as a bulk effect of a system of interacting fermions. Yet it was computed considering non-interacting electrons. This apparent inconsistency is solved by a duality argument: a system of interacting electrons of a single species in a magnetic field at half-filling is visualized as a bidimensional gas of electrons at zero magnetic field which exhibits QHE. This state is pictured as an electrical electron-vortex dipole [34].

In this fashion, the findings of AS, equation (6), which we have proved to be correct, describe a very interesting physical system. Their misinterpretations, presumably stated to enforce their results to the IQHE, cannot be correct. Hence, the findings of papers based on AS reasoning, [14, 15], specially in the description of the FQHE, need further revision. A key observation for the description of QHE for systems of relativistic fermions is the non-invariance of the mass terms under spatial and time reflections. This becomes even more relevant in systems like graphene, where a plethora of mass terms with a variety of spacetime transformation properties can be considered [39]. The complete identification of such mass terms in the dynamics of QHE in graphene is in progress.

Acknowledgments

Appendix

The Hall conductivity is defined as the linear part of the current–current correlation function [4] :

$$\sigma_{xy} = \frac{1}{3!}\epsilon_{\mu\rho\lambda}\left.\frac{\partial}{\partial q^\lambda}\Pi^{\mu\rho}(q^2)\right|_{q^2=0}. \tag{A.1}$$

From the tensor structure of the vacuum polarization (32), we see that the symmetric part of the vacuum polarization, $\Pi_e^{\mu\nu}$, vanishes upon contraction with the Levi-Civita symbol. Thus

$$\sigma_{xy} = \frac{1}{3!}\epsilon_{\mu\rho\lambda}\left.\frac{\partial}{\partial q^\lambda}(im\epsilon^{\mu\rho\eta}q_\eta\Pi_o(q^2))\right|_{q^2=0}$$

$$= im\Pi_o(0) + \frac{im}{3!}q_\eta\left.\frac{\partial}{\partial q^\eta}\Pi_o(q^2)\right|_{q^2=0}. \tag{A.2}$$

Now, from the vacuum polarization (31), we have that

$$\Pi_o(q^2) = \frac{-ie^2}{4\pi^3}\int d^3k\,\frac{1}{[k^2-m^2][(k+q)^2-m^2]}$$

$$= \frac{-ie^2}{4\pi\sqrt{q^2}}\ln\left(\frac{2m+\sqrt{q^2}}{2m-\sqrt{q^2}}\right). \tag{A.3}$$

From this expression, it is not difficult to see that

$$\Pi_o(0) = \frac{-ie^2}{4\pi m}, \qquad q^2 \frac{\partial}{\partial q^2} \Pi_o(q^2)\bigg|_{q=0} = 0, \tag{A.4}$$

and thus

$$\sigma_{xy} = im\Pi_o(0) = \frac{e^2}{4\pi}. \tag{A.5}$$

Comparing with (1), we again obtain $\nu = -1/2$. Although we have used an irreducible representation of the Dirac matrices in this calculation, the same result per fermion species holds with a reducible representation, provided we consider the term $m_0 \bar{\psi} \tau \psi$. Furthermore, note that $\Pi_o(0)$ is precisely the coefficient of the CST in (30), thus the non-vanishing value of filling factor is related to the non-vanishing of this coefficient, as we stated earlier.

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